CANONICAL 2-FORMS ON THE MODULI SPACE OF RIEMANN SURFACES

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ABSTRACT. As was shown by Harer [14] [15], the second homology of $\mathbb{M}_g$, the moduli space of compact Riemann surfaces of genus g, is of rank 1, provided $g \geq 3$. This means there exists a nontrivial second de Rham cohomology class on $\mathbb{M}_g$ which is unique up to a constant factor. But several canonical 2-forms on the moduli space have been constructed in various geometric contexts, and they differ from each other. In this article we review some constructions of such canonical 2-forms in order to provide material for future research on the "secondary geometry" of the moduli space $\mathbb{M}_g$.

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1. INTRODUCTION

Let $g \geq 2$ be an integer. The moduli space of compact Riemann surfaces of genus g, $\mathbb{M}_g$, is the quotient space of Teichmüller space $\mathcal{T}_g$ by the natural action of the mapping class group $\mathcal{M}_g$, $\mathbb{M}_g = \mathcal{T}_g/\mathcal{M}_g$. Since Teichmüller space is contractible, the real cohomology of the

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mapping class group is isomorphic to that of the moduli space. As was shown by Harer [14] [15], the second homology of $\mathbb{M}_g$ is of rank 1 if $g \geq 3$. This means there exists a nontrivial second de Rham cohomology class on $\mathbb{M}_g$ which is unique up to a constant factor. But several canonical 2-forms on the moduli space have been constructed in various geometric contexts, and they differ from each other. In this article we review some constructions of such canonical 2-forms in order to provide material for future research on the "secondary geometry" of the moduli space $\mathbb{M}_g$.

The signature of the total space of a fiber bundle is not necessarily equal to the product of the signatures of the base space and the fiber. The first example for this phenomenon was given by Kodaira [27] and Atiyah [6], who constructed a certain branched covering space of the product of two compact Riemann surfaces. The covering space has non-zero signature, while the signature of any compact Riemann surface is zero. We may regard the covering space as a family of compact Riemann surfaces parametrized by a compact Riemann surface, so that it defines a non-trivial 2-cycle on the space $\mathbb{M}_g$. As was formulated by Meyer [30] [31], the signature of the total space of a family of compact Riemann surfaces defines a non-trivial 2-cocycle of the mapping class group $\mathcal{M}_g$ and this provides a non-trivial cohomology class of degree 2 on the space $\mathbb{M}_g$. Nowadays this cocycle is called the Meyer cocycle and it has been playing an essential role in the topological study of fibered complex surfaces. See [4] and [5] for details.

The first and the second Betti numbers of the space $\mathbb{M}_g$, or equivalently, those of the group $\mathcal{M}_g$, are given by

$$b_1(\mathbb{M}_g) = 0, \qquad \text{[41] [45] [14, p.223]} \tag{1.1}$$

$$b_2(\mathbb{M}_g) = 1, \quad \text{if } g \geq 3. \quad \text{[14] [15]} \tag{1.2}$$

For alternative computations of $b_2(\mathbb{M}_g)$, see [2] [28] [44]. The group $H^2(\mathbb{M}_g; \mathbb{R})$ is generated by the cohomology class of the Meyer cocycle. In the case $g = 2$ we have $b_2(\mathbb{M}_2) = 0$ because of Igusa's result $\mathbb{M}_2 = \mathbb{C}^3/(\mathbb{Z}/5) \simeq *$ [12].

Mumford [42] and Morita [33] independently introduced a series of cohomology classes $e_n = (-1)^{n+1}\kappa_n \in H^{2n}(\mathbb{M}_g)$, $n \geq 1$, the Morita-Mumford classes or the tautological classes. They are defined as follows. Let $\pi : \mathbb{C}_g \to \mathbb{M}_g$ be the universal family of compact Riemann surfaces of genus g. The relative tangent bundle of the map π, $T_{\mathbb{C}_g/\mathbb{M}_g}$, the kernel of the differential $d\pi : T\mathbb{C}_g \to \pi^* T\mathbb{M}_g$, is a complex line V-bundle over $\mathbb{C}_g$. The n-th Morita-Mumford class $e_n = (-1)^{n+1}\kappa_n$, $n \geq 1$, is defined to be the integral of the $(n+1)$-st power of the Chern

class of the bundle $T^{\times}_{\mathbb{C}_g/\mathbb{M}_g}$ along the fiber

$$e_n = (-1)^{n+1}\kappa_n = \int_{\text{fiber}} c_1(T^{\times}_{\mathbb{C}_g/\mathbb{M}_g})^{n+1} \in H^{2n}(\mathbb{M}_g). \tag{1.3}$$

The first one $e_1 = \kappa_1$ is 3 times the cohomology class of the Meyer cocycle. As was proved by Morita [34] and Miller [32], the Morita-Mumford classes are algebraically independent in the stable range $* < \frac{2}{3}g$ [16] of the cohomology algebra $H^*(\mathbb{M}_g;\mathbb{R})$. Their proofs generalize the construction of Kodaira and Atiyah. In 2002 Madsen and Weiss [29] proved that the cohomology algebra $H^*(\mathbb{M}_g;\mathbb{R})$ in the stable range is generated by the Morita-Mumford classes.

From the results (1.1) and (1.2) the simplest non-trivial cohomology classes on $\mathbb{M}_g$ are of degree 2, and they are unique up to a constant factor. But several 2-forms on $\mathbb{M}_g$, or equivalently $\mathcal{M}_g$-equivariant 2-forms on Teichmüller space $\mathcal{T}_g$, have been canonically constructed in various geometric contexts.

From the uniformization theorem any compact Riemann surface C of genus $g \geq 2$ admits a unique hyperbolic metric. The volume form of the hyperbolic metric defines the Weil-Petersson pairing on the cotangent space $T^*_{[C]}\mathbb{M}_g$ involved with no addditional information. As was shown by Wolpert [48] the Weil-Petersson-Kähler form ω_{WP} represents the first Morita-Mumford class e_1. Thus we obtain a canonical 2-form representing e_1.

The period map is a canonical map defined on Teichmüller space into the Siegel upper halfspace $\mathfrak{H}_g$. We have a canonical 2-form on $\mathfrak{H}_g$ whose pullback represents the class e_1 on the moduli space $\mathbb{M}_g$.

We have another canonical metric on a compact Riemann surface. A natural Hermitian product on the space of holomorphic 1-forms defines the volume form B in 5.3 which induces a Hermitian metric on the Riemann surface. The Arakelov-Green function is derived from the volume form B. As will be stated in §7 and §8, a higher analogue of the period map is constructed and yields other canonical 2-forms representing e_1. These forms are closely related to the volume form B.

All of them differ from each other. As to 2-forms representing non-trivial cohomology classes of degree 2 on the moduli space $\mathbb{M}_g$, the term ‘canonical’ does not imply ‘unique’. The difference of such forms should induce some secondary object on the moduli space $\mathbb{M}_g$. Assume $g \geq 3$. If we have two real $(1,1)$-forms ψ_1 and ψ_2 on $\mathbb{M}_g$ representing e_1, then there exists a real-valued function $f \in C^\infty(M;\mathbb{R})$ such that $\psi_2 - \psi_1 = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial} f$. Such a function f is unique up to a constant. See Lemma 8.1. This function captures the difference between these two

forms, so that it should describe a certain relation between the two geometric contexts behind these forms.

In this article we review some constructions of canonical 2-forms. In §2 we give a short review on the cotangent spaces of moduli spaces. They are naturally isomorphic to some spaces of quadratic differentials. In §3 we take a quick glance at the Weil-Petersson Kähler form, which is related to the Virasoro cocycle through the Krichever construction. The most classical 2-form on $\mathbb{M}_g$ is the pullback of the first Chern form on the Siegel upper halfspace $\mathfrak{H}_g$ by the period map Jac, or equivalently the first Chern form of the Hodge bundle on $\mathbb{M}_g$. We explain this form in §§4 and 5. The Hodge bundle yields all the odd Morita-Mumford classes but not the even ones. We can obtain other canonical differential forms on the moduli space representing all the Morita-Mumford class e_i, $i \geq 1$, through a higher analogue of the period map, and this is described in §§6 and 7. Among them some 2-forms seem to be related to Arakelov geometry, as will be discussed in §8.

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2. The cotangent space of the moduli space

Let C be a compact Riemann surface of genus $g \geq 2$, P_0 a point on C. Then we denote by $H^q(C; aK + bP_0)$, $q = 0, 1$, and $a, b \in \mathbb{Z}$, the q-th cohomology group $H^q(C; \mathcal{O}_C(T^*C^{\otimes a} \otimes [P_0]^{\otimes b}))$. Moreover we denote by $\Omega^q(C)$ the complex-valued q-currents on C for $0 \leq q \leq 2$. The Hodge $*$-operator $* : (T^*_{\mathbb{R}}C) \otimes \mathbb{C} \to (T^*_{\mathbb{R}}C) \otimes \mathbb{C}$ on the cotangent bundle of C depends only on the complex structure of C. The $-\sqrt{-1}$-eigenspace is the holomorphic cotangent bundle T^*C, and the $\sqrt{-1}$-eigenspace is the antiholomorphic cotangent bundle $\overline{T^*C}$. The operator $*$ decomposes the space $\Omega^1(C)$ into the $\pm\sqrt{-1}$-eigenspaces

$$\Omega^1(C) = \Omega^{1,0}(C) \oplus \Omega^{0,1}(C),$$

where $\Omega^{1,0}(C)$ is the $-\sqrt{-1}$-eigenspace and $\Omega^{0,1}(C)$ the $\sqrt{-1}$-eigenspace. Throughout this article we denote by φ' and φ'' the $(1,0)$- and the $(0,1)$-parts of $\varphi \in \Omega^1(C)$, respectively, i.e.,

$$\varphi = \varphi' + \varphi'', \quad *\varphi = -\sqrt{-1}\varphi' + \sqrt{-1}\varphi''.$$

If φ is harmonic, then φ' is holomorphic and φ'' anti-holomorphic.

The Kodaira-Spencer map gives a natural isomorphism

$$T_{[C]}\mathbb{M}_g = H^1(C; -K). \tag{2.1}$$

To look at the isomorphism (2.1) more explicitly, consider a C^∞ family of compact Riemann surfaces C_t, $t \in \mathbb{R}$, $|t| \ll 1$, with $C_0 = C$. The family $\{C_t\}$ is trivial as a C^∞ fiber bundle over an interval near $t = 0$, so that we have a C^∞ family of C^∞ diffeomorphisms $f^t : C \to C_t$ with $f^0 = 1_C$. In general, if $\bigcirc = \bigcirc_t$ is a "function" in $t \in \mathbb{R}$, $|t| \ll 1$, then we write simply

$$\dot{\bigcirc} = \frac{d}{dt}\Big|_{t=0} \bigcirc_t .$$

For example, we denote

$$\dot{\mu} = \frac{d}{dt}\Big|_{t=0} \mu(f^t).$$

Here $\mu(f^t)$ is the complex dilatation of the diffeomorphism f^t. Let z_1 be a complex coordinate on C, and ζ_1 on C_t. The complex dilatation $\mu(f^t)$ is defined locally by

$$\mu(f^t) = \mu(f^t)(z_1)\frac{d}{dz_1} \otimes d\overline{z_1} = \frac{(\zeta_1 \circ f^t)_{\overline{z_1}}}{(\zeta_1 \circ f^t)_{z_1}} \frac{d}{dz_1} \otimes d\overline{z_1},$$

which does not depend on the choice of the coordinates z_1 and ζ_1. The Dolbeault cohomology class $[\dot{\mu}] \in H^1(C; -K)$ is exactly the tangent vector $\frac{d}{dt}\big|_{t=0}[C_t] \in T_{[C]}\mathbb{M}_g$.

We define a linear operator $S = S[\dot{\mu}] : \Omega^1(C) \to \Omega^1(C)$ by

$$S(\varphi) = S(\varphi') + S(\varphi'') := -2\varphi'\dot{\mu} - 2\varphi''\overline{\dot{\mu}},$$

for $\varphi = \varphi' + \varphi''$, $\varphi' \in \Omega^{1,0}(C)$, $\varphi'' \in \Omega^{0,1}(C)$. From straightforward computation we have

$$\dot{*} = *S = -S* : \Omega^1(C) \to \Omega^1(C). \tag{2.2}$$

By Serre duality we have a natural isomorphism

$$T^*_{[C]}\mathbb{M}_g = H^0(C; 2K). \tag{2.3}$$

The space $H^0(C; 2K)$ consists of the holomorphic quadratic differentials on C. For any holomorphic quadratic differential q the covariant tensor $q\dot{\mu}$ can be regarded as a $(1,1)$-form on C. The integral $\int_C q\dot{\mu}$ is just the value of the covector q at the tangent vector $[\dot{\mu}] = \frac{d}{dt}\big|_{t=0}[C_t]$.

Let $\mathbb{C}_g$ denote the moduli space of pointed compact Riemann surfaces (C, P_0) of genus g with $P_0 \in C$. The forgetful map $\pi : \mathbb{C}_g \to \mathbb{M}_g$, $[C, P_0] \mapsto [C]$, can be interpreted as the universal family of compact Riemann surfaces on the moduli space $\mathbb{M}_g$. We identify

$$T_{[C,P_0]}\mathbb{C}_g = H^1(C; -K - P_0), \quad \text{and} \quad T^*_{[C,P_0]}\mathbb{C}_g = H^0(C; 2K + P_0) \tag{2.4}$$

in a way similar to the space $\mathbb{M}_g$.

The relative tangent bundle of the forgeful map π with the zero section deleted

$$T^{\times}_{\mathbb{C}_g/\mathbb{M}_g} = T_{\mathbb{C}_g/\mathbb{M}_g} \setminus (\text{zero section})$$

can be interpreted as the moduli space of triples (C, P_0, v) of genus g. Here C is a compact Riemann surface of genus g, $P_0 \in C$, and $v \in T_{P_0}C \setminus \{0\}$. Similarly the space of quadratic differentials $H^0(C; 2K + 2P_0)$ is identified with the cotangent space of $T^{\times}_{\mathbb{C}_g/\mathbb{M}_g}$

$$T^*_{[C,P_0,v]} T^{\times}_{\mathbb{C}_g/\mathbb{M}_g} = H^0(C; 2K + 2P_0). \tag{2.5}$$

Moreover this space is closely related to Ehresmann connections on the bundle $T_{\mathbb{C}_g/\mathbb{M}_g}$. In general, let $\varpi : L \to M$ be a holomorphic line bundle over a complex manifold M, and $L^\times$ the total space with the zero section deleted $L^\times = L \setminus (\text{zero section})$. We denote by R_a the right action of $a \in \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ on the space $L^\times$, and by Z the vector field on $L^\times$ generated by the action R_a

$$Z := \left.\frac{d}{dt}\right|_{t=0} R_{e^t}.$$

An Ehresmann connection A (of type $(1,0)$) on the bundle L is a $(1,0)$-form on the space $L^\times$ with the conditions

$$\begin{aligned} A(Z) &= 1, \quad \text{and} \\ {R_{e^t}}^* A &= A, \quad \forall t \in \mathbb{R} \end{aligned}$$

[7][26]. In other words, it is a splitting of the extension of holomorphic vector bundles over M

$$0 \to T^*M \xrightarrow{\varpi^*} (T^*L^\times)/\mathbb{C}^\times \xrightarrow{Z} \mathbb{C} \to 0.$$

Then there exists a unique $(1,1)$-form $c_1(A)$ on M such that $\frac{\sqrt{-1}}{2\pi} dA = \varpi^* c_1(A)$. The form $c_1(A)$ is, by definition, the Chern form of the connection A and represents the first Chern class of the line bundle L

$$[c_1(A)] = c_1(L) \in H^2(M; \mathbb{R}).$$

Now we let $M = \mathbb{C}_g$ and $L = T_{\mathbb{C}_g/\mathbb{M}_g}$. By straightforward computation we have a natural commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T^*_{[C,P_0]}M & \xrightarrow{\varpi^*} & ((T^*L^\times)/\mathbb{C}^\times)_{[C,P_0]} & \xrightarrow{Z} & \mathbb{C} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & H^0(C; 2K + P_0) & \longrightarrow & H^0(C; 2K + 2P_0) & \xrightarrow{2\pi\sqrt{-1}\,\mathrm{Res}_{P_0}} & \mathbb{C} & \longrightarrow & 0
\end{array}$$

Here $\mathrm{Res}_{P_0} : H^0(C; 2K + 2P_0) \to \mathbb{C}$ is the residue map of quadratic differentials at P_0 defined by

$$\mathrm{Res}_{P_0}(q_{-2}z^{-2} + q_{-1}z^{-1} + q_0 + q_1 z^1 + \cdots)dz^{\otimes 2} = q_{-2},$$

where z is a complex coordinate centered at P_0. It is easy to check q_{-2} does not depend on the choice of the coordinate z. Consequently any C^∞ family $q = \{q(C, P_0)\}_{[C,P_0]\in\mathbb{C}_g}$, $q(C, P_0) \in H^0(C; 2K + 2P_0)$ of quadratic differentials parametrized by the space $\mathbb{C}_g$ satisfying the condition $\mathrm{Res}_{P_0}\, q(C, P_0) = \frac{1}{2\pi\sqrt{-1}}$ for any $[C, P_0] \in \mathbb{C}_g$ corresponds to an Ehresmann connection on the relative tangent bundle $T_{\mathbb{C}_g/\mathbb{M}_g}$. The $(1, 1)$ form $\frac{\sqrt{-1}}{2\pi}\overline{\partial}q$ on the space $\mathbb{C}_g$ represents the first Chern class of the bundle $T_{\mathbb{C}_g/\mathbb{M}_g}$

$$\frac{\sqrt{-1}}{2\pi}[\overline{\partial}q] = c_1(T_{\mathbb{C}_g/\mathbb{M}_g}) \in H^2(\mathbb{C}_g; \mathbb{R}) \tag{2.6}$$

[20].

3. The Weil-Petersson Kähler form

As was shown in §2 the cotangent space of the moduli space $\mathbb{M}_g$ at $[C]$ is naturally isomorphic to the space of holomorphic quadratic differentials, $H^0(C; 2K)$. Let dvol denote the hyperbolic volume form on the Riemann surface C. It is regarded as a Hermitian metric on the relative tangent bundle $T_{\mathbb{C}_g/\mathbb{M}_g}$. For any two differentials $q_1, q_2 \in H^0(C; 2K)$ the Weil-Petersson pairing $\langle q_1, q_2\rangle_{\mathrm{WP}}$ is defined by the integral

$$\langle q_1, q_2\rangle_{\mathrm{WP}} = \int_C q_1\overline{q_2}/\mathrm{dvol}\,.$$

Here $q_1\overline{q_2}/\mathrm{dvol}$ is regarded as a $(1, 1)$-form on C. The pairing induces a Hermitian metric on the moduli space $\mathcal{M}_g$, the Weil-Petersson metric. Ahlfors [1] proved it is Kähler. See [10] for an alternative gauge-theoretic proof. Let ω_{WP} denote the Kähler form of the Weil-Petersson metric.

Now recall the original definition of the i-th Morita-Mumford classs $e_i = (-1)^{i+1}\kappa_i$, $i \geq 1$ [42] [33]. It is defined to be the integral along the fiber of the $(i + 1)$-st power of the first Chern class of the relative tangent bundle $T_{\mathbb{C}_g/\mathbb{M}_g}$

$$e_i = (-1)^{i+1}\kappa_i = \int_{\mathrm{fiber}} c_1(T_{\mathbb{C}_g/\mathbb{M}_g})^{i+1} \in H^{2i}(\mathbb{M}_g). \tag{3.1}$$

It is one of the most orthodox ways to obtain differrential forms representing the Morita-Mumford classes to take the integral of powers of the hyperbolic Chern form of the relative tangent bundle $T_{\mathbb{C}_g/\mathbb{M}_g}$ along

the fiber. This was carried out by Wolpert [48]. He computed the Chern form $c_1^{\text{hyperbolic}}(T_{\mathbb{C}_g/\mathbb{M}_g})$ of the hyperbolic metric explicitly, and he proved

$$\int_{\text{fiber}} c_1^{\text{hyperbolic}}(T_{\mathbb{C}_g/\mathbb{M}_g})^2 = \frac{1}{2\pi^2}\omega_{\mathrm{WP}} \tag{3.2}$$

as differential forms on the moduli space $\mathbb{M}_g$. As a corollary we have

$$\frac{1}{2\pi^2}[\omega_{\mathrm{WP}}] = e_1 \in H^2(\mathbb{M}_g;\mathbb{R}).$$

Furthermore Wolpert [49] gave a description of the Weil-Petersson Kähler form in terms of the Fenchel-Nielsen coordinates (τ_j, ℓ_j), $1 \leq j \leq 3g-3$, for any pants decomposition of the surface

$$\omega_{\mathrm{WP}} = \sum d\ell_i \wedge d\tau_i. \tag{3.3}$$

Here ℓ_j denotes the geodesic length of each simple closed curve in the decomposition, and $\tau_j \in \mathbb{R}$ the hyperbolic displacement parameter. Penner [43] described explicitly the pullback of ω_{WP} to the decorated Teichmüller space. Goldman [11] generalized the Weil-Petersson geometry to the space of surface group representations in a reductive Lie group.

Now we consider the Lie algebra $\mathbf{d}$ of complex analytic vector fields on the punctured disk $\{z \in \mathbb{C};\ 0 < |z| < \epsilon\}$, $0 < \epsilon \ll 1$. The 2-cochain vir on $\mathbf{d}$ defined by

$$\begin{aligned} \mathrm{vir}\left(f_1(z)\frac{d}{dz}, f_2(z)\frac{d}{dz}\right) &:= \frac{1}{2\pi\sqrt{-1}}\oint_{|z|=1} \det\begin{pmatrix} f_1'(z) & f_2'(z) \\ f_1''(z) & f_2''(z)\end{pmatrix} dz \\ &= \frac{\sqrt{-1}}{2\pi}\oint_{|z|=1} \det\begin{pmatrix} f_1(z) & f_2(z) \\ f_1'''(z) & f_2'''(z)\end{pmatrix} dz \end{aligned}$$

is a cocycle and it is called the Virasoro cocycle. Its cohomology class generates the second Lie algebra cohomology group $H^2(\mathbf{d}) = \mathbb{C}$.

Arbarello, De Concini, Kac and Procesi [3] established an isomorphism of $H^2(\mathbf{d})$ onto the second cohomology group of $\mathbb{M}_g$

$$\nu : H^2(\mathbf{d}) \overset{\cong}{\to} H^2(\mathbb{M}_g;\mathbb{C}) \tag{3.4}$$

induced by the Krichever construction.

For a local coordinate z on a Riemann surface one can define a local differential operator, or a local complex analytic Gel'fand-Fuks 1-cocycle with values in quadratic differentials by

$$\nabla_2^{d/dz} : f(z)\frac{d}{dz} \mapsto \frac{1}{6}f'''(z)(dz)^{\otimes 2}$$

[19, p.666]. The cocycle $\nabla_2^{d/dz}$ is equivalent to a projective structure. In fact, if w is another coordinate, then

$$\nabla_2^{d/dw} X - \nabla_2^{d/dz} X = \mathcal{L}_X \left(\{w, z\} (dz)^{\otimes 2} \right)$$

for any local complex analytic vector field X. Here $\{w, z\}$ denotes the Schwarzian derivative. In particular, the hyperbolic structure on a (hyperbolic) Riemann surface defines a global operator $\nabla_2^{\text{hyperbolic}}$.

The Krichever construction relates the 2-cocycle vir with the operator $\nabla_2^{\text{hyperbolic}}$. By straightforward computation using the Bers embedding we have

$$\overline{\partial} \nabla_2^{\text{hyperbolic}} = 8\omega_{\text{WP}} \tag{3.5}$$

as $(1,1)$-forms on the moduli space $\mathbb{M}_g$. This result, the first variation of the hyperbolic structure coincides with ω_{WP}, was first proved by Zograf and Takhtajan [50, p.310].

4. The first Chern form on the Siegel upper halfspace

The Hodge bundle $\Lambda_{\mathbb{M}_g}$ is defined to be the holomorphic vector bundle on $\mathbb{M}_g$ whose fiber over $[C]$ is the space of holomorphic 1-forms on C

$$\Lambda_{\mathbb{M}_g} = \coprod_{[C] \in \mathbb{M}_g} H^0(C; K).$$

We write simply c_1 for the first Chern class of $\Lambda_{\mathbb{M}_g}$

$$c_1 = c_1(\Lambda_{\mathbb{M}_g}) \in H^2(\mathbb{M}_g; \mathbb{R}).$$

The bundle $\Lambda_{\mathbb{M}_g}$ comes from a symplectic equivariant vector bundle on the Siegel upper halfspace $\mathfrak{H}_g$. In fact, the space $\mathfrak{H}_g$ can be identified with the space of almost complex structures J on the real $2g$-dimensional symplectic vector space $(\mathbb{R}^{2g}, \cdot)$ with the conditions

$$Jx \cdot Jy = x \cdot y, \quad \forall x, \forall y \in \mathbb{R}^{2g},$$
$$x \cdot Jx > 0, \quad \forall x \in \mathbb{R}^{2g} \setminus \{0\}.$$

We have a holomorphic vector bundle $E'_{\mathfrak{H}_g}$ on $\mathfrak{H}_g$ whose fiber over J is the $-\sqrt{-1}$-eigenspace of J. We have a natural isomorphism of vector bundles

$$T^*\mathfrak{H}_g = \operatorname{Sym}^2 E'_{\mathfrak{H}_g}. \tag{4.1}$$

For each Riemann surface C the Hodge $*$-operator on the 1-forms induces such an almost complex structure on the space of real harmonic 1-forms. This induces a holomorphic map $\mathrm{Jac} : \mathbb{M}_g \to \mathfrak{H}_g / Sp_{2g}(\mathbb{Z})$ known as the period map in the classical context. The pullback of $E'_{\mathfrak{H}_g}$ by the map Jac is exactly the Hodge bundle $\Lambda_{\mathbb{M}_g}$.

Thus the cohomology class c_1 can be regarded as an integral cohomology class of the Siegel modular group $Sp_{2g}(\mathbb{Z})$, $c_1 \in H^2(Sp_{2g}(\mathbb{Z});\mathbb{Z})$. Meyer [30] proved that the cohomology class of the Meyer cocycle is equal to $4c_1 \in H^2(Sp_{2g}(\mathbb{Z});\mathbb{Z})$. From the Grothendieck-Riemann-Roch formula, or equivalently the Atiyah-Singer index theorem for families, it follows that

$$\frac{1}{12}e_1 = c_1 \in H^2(\mathbb{M}_g;\mathbb{R}). \tag{4.2}$$

To describe a canonical 2-form representing $c_1(E'_{\mathfrak{H}_g})$ we consider the quotient vector bundle $E''_{\mathfrak{H}_g} := (\mathfrak{H}_g \times \mathbb{C}^{2g})/E'_{\mathfrak{H}_g}$, and the family of projections $\pi = \{\pi_J\}_{J\in\mathfrak{H}_g}$ on $\mathbb{C}^{2g}$, $\pi_J := \frac{1}{2}(1-\sqrt{-1}J)$, parametrized by $\mathfrak{H}_g$. Then $\{\pi_J \circ d\}_{J\in\mathfrak{H}_g}$ is a covariant derivative ∇ of type $(1,0)$ on the bundle $E''_{\mathfrak{H}_g} \cong \coprod_{J\in\mathfrak{H}_g} \operatorname{Image}\pi_J$, whose curvature form R^∇ is given by

$$R^\nabla = \pi(\partial\pi)(\overline{\partial}\pi). \tag{4.3}$$

The 2-form $c_1(\nabla)$ defined by $c_1(\nabla) = \frac{\sqrt{-1}}{2\pi}\operatorname{trace} R^\nabla$ represents $c_1(E'_{\mathfrak{H}_g})$. Let $J_\alpha(t) \in \mathfrak{H}_g$, $|t| \ll 1$, $\alpha = 1,2$, be C^∞ paths on $\mathfrak{H}_g$ with $J_1(0) = J_2(0) = J$. Then, one can compute

$$c_1(\nabla)_J = \frac{1}{8\pi}\operatorname{trace}(\dot{J}_1\, J\, \dot{J}_2). \tag{4.4}$$

In the next section we prove Rauch's variational formula to obtain the pullback of $c_1(\nabla)_J$ by the period map Jac explicitly.

5. Rauch's variational formula

Rauch's variational formula describes the differential of the period map Jac. Let C be a compact Riemann surface of genus g. We denote by H the real first homology group $H_1(C;\mathbb{R})$. Consider the map $H^* = H^1(C;\mathbb{R}) \to \Omega^1(C)$ assigning to each cohomology class the harmonic 1-form representing it. The map can be regarded as an H-valued 1-form $\omega_{(1)} \in \Omega^1(C)\otimes H$.

Let $\{X_i, X_{g+i}\}_{i=1}^g$ be a symplectic basis of $H_\mathbb{C} = H_1(C;\mathbb{C})$

$$X_i \cdot X_{g+j} = \delta_{ij}, \quad X_i \cdot X_j = X_{g+i}\cdot X_{g+j} = 0, \quad 1 \le i,j \le g,$$

and $\{\xi_i, \xi_{g+i}\}_{i=1}^g \subset \Omega^1(C)$ the basis of the harmonic 1-forms dual to $\{X_i, X_{g+i}\}_{i=1}^g$. Then we have

$$\omega_{(1)} = \sum_{i=1}^g \xi_i X_i + \xi_{g+i}X_{g+i} \in \Omega^1(C)\otimes H_\mathbb{C}.$$

In particular, if $\{\psi_i\}_{i=1}^g \subset H^0(C;K)$ is an orthonormal basis

$$\frac{\sqrt{-1}}{2}\int_C \psi_i \wedge \overline{\psi_j} = \delta_{ij}, \quad 1 \leq i,j \leq g, \tag{5.1}$$

then we obtain

$$\omega_{(1)} = \sum_{i=1}^g \psi_i Y_i + \overline{\psi_i Y_i}, \tag{5.2}$$

where $\{Y_i, Y_{g+i}\}_{i=1}^g \subset H_{\mathbb{C}}$ is the dual basis of the symplectic basis $\{[\psi_i], \frac{\sqrt{-1}}{2}[\overline{\psi_i}]\}_{i=1}^g$ of $H^*_{\mathbb{C}} = H^1(C;\mathbb{C})$. Since the complete linear system of the canonical divisor on the complex algebraic curve C has no basepoint, the 2-form

$$B = \frac{1}{2g}\omega_{(1)} \cdot \omega_{(1)} = \frac{\sqrt{-1}}{2g}\sum_{i=1}^g \psi_i \wedge \overline{\psi_i} \in \Omega^2(C) \tag{5.3}$$

is a volume form on C.

Now we recall the Hodge decomposition of the 1-forms on C. We have an exact sequence

$$0 \to \mathbb{C} \to \Omega^0(C) \overset{d*d}{\to} \Omega^2(C) \overset{\int_C}{\to} \mathbb{C} \to 0.$$

The vector space $\mathbb{C}$ on the left side means the constant functions. A Green operator $\Psi : \Omega^2(C) \to \Omega^0(C)$ is a linear map satisfying the property

$$d * d\Psi\Omega = \Omega$$

for any $\Omega \in \Omega^2(C)$ with $\int_C \Omega = 0$. In this article we use two sorts of Green operators $\widehat{\Phi} = \widehat{\Phi}_C$ and $\Phi = \Phi^{(C,P_0)}$. The former is characterized by the conditions

$$d * d\widehat{\Phi}(\Omega) = \Omega - (\int_C \Omega)B \quad \text{and} \quad \int_C \widehat{\Phi}(\Omega)B = 0 \tag{5.4}$$

for any $\Omega \in \Omega^2(C)$. Let $\delta_{P_0} : C^\infty(C) \to \mathbb{C}$, $f \mapsto f(P_0)$, be the delta current on C at the point P_0. We define the latter Φ to be a linear map with values in $\Omega^0(C)/\mathbb{C}$ instead of $\Omega^0(C)$. Then the operator $d\Phi : \Omega^2(C) \to \Omega^1(C)$ makes sense, and the operator Φ is defined by the condition

$$d * d\Phi\Omega = \Omega - \left(\int_C \Omega\right)\delta_{P_0}$$

for any $\Omega \in \Omega^2(C)$.

Any Green operator Ψ induces the Hodge decomposition of the 1-currents

$$\varphi = \mathcal{H}\varphi + d\Psi d * \varphi + *d\Psi d\varphi \tag{5.5}$$

for any $\varphi \in \Omega^1(C)$, where $\mathcal{H} : \Omega^1(C) \to \Omega^1(C)$ is the harmonic projection on the 1-currents on C.

In the setting of §2 the first variation of $\omega_{(1)}$ is given by

$$\dot{\omega_{(1)}} = -d\Psi d * S\omega_{(1)}. \tag{5.6}$$

In fact, differentiating $d * \omega_{(1)} = 0$, we get

$$d * \dot{\omega_{(1)}} = -d \dot{*} \omega_{(1)} = -d * S\omega_{(1)}.$$

Since ${f^t}^*\omega_{(1)}$ is cohomologous to $\omega_{(1)}$, we have some function u such that $\dot{\omega_{(1)}} = du$. Hence from (5.5) we obtain

$$\dot{\omega_{(1)}} = d\Psi d * \dot{\omega_{(1)}} = -d\Psi d * \omega_{(1)},$$

as was to be shown.

Theorem 5.1 (Rauch)**.** *The diagram*

$$\begin{array}{ccc} T^*_{[C]}\mathbb{M}_g & \xleftarrow{(d\,\mathrm{Jac})^*} & T^*_{[\mathrm{Jac}(C)]}\mathfrak{H}_g/Sp_{2g}(\mathbb{Z}) \\ \| & & \| \\ H^0(C;2K) & \xleftarrow{2\sqrt{-1}(\textit{multiplication})} & \mathrm{Sym}^2 H^0(C;K). \end{array}$$

commutes. Here the lower horizontal arrow maps $\psi_1 \otimes \psi_2$ to the quadratic differential $2\sqrt{-1}\psi_1\psi_2$ for any 1-forms ψ_1 and $\psi_2 \in H^0(C;K)$.

Proof. The integral $\int_C *\omega_{(1)} \wedge \omega_{(1)} \in H \otimes H = H^* \otimes H = \mathrm{Hom}(H,H)$ coincides with the almost complex structure on $H = H_1(C;\mathbb{R})$ induced by the Hodge $*$-operator. Since $\omega_{(1)}$ is harmonic and $\dot{\omega_{(1)}}$ is d-exact by (5.6), we have

$$\int_C *\omega_{(1)} \wedge \dot{\omega_{(1)}} = -\int_C \omega_{(1)} \wedge *\dot{\omega_{(1)}} = 0.$$

Hence

$$\begin{aligned} \left(\textstyle\int_C *\omega_{(1)} \wedge \omega_{(1)}\right)^{\cdot} &= \textstyle\int_C \dot{*}\omega_{(1)} \wedge \omega_{(1)} = \int_C (*S\omega_{(1)}) \wedge \omega_{(1)} \\ &= 2\sqrt{-1}\textstyle\int_C {\omega_{(1)}}'{\omega_{(1)}}'\dot{\mu} - 2\sqrt{-1}\overline{\left(\int_C {\omega_{(1)}}'{\omega_{(1)}}'\dot{\mu}\right)}. \end{aligned}$$

This proves the theorem. □

Substituting the theorem into the formula (4.4) we have

Corollary 5.2.

$$\mathrm{Jac}^* c_1(\nabla) = \frac{1}{8\pi\sqrt{-1}} \sum_{i,j=1}^{g} \psi_i\psi_j \otimes \overline{\psi_i\psi_j} \in T^*_{[C]}\mathbb{M}_g \otimes \overline{T^*_{[C]}\mathbb{M}_g}.$$

Here $\{\psi_i\}_{i=1}^g \subset H^0(C;K)$ *is any orthonormal basis (5.1).*

The elementary polynomials $\sigma_1, \dots, \sigma_g$ in indeterminates $x_1, \dots, x_g$ are given by $\prod_{i=1}^g (t - x_i) = t^g + \sum_{k=1}^g (-1)^k \sigma_k t^{g-k}$. The equation $\sum_{i=1}^g {x_i}^m = s_m(\sigma_1, \dots, \sigma_g)$ defines the m-th Newton polynomial s_m. The m-th Newton class of the Hodge bundle $\Lambda = \Lambda_{\mathbb{M}_g}$ is defined by

$$s_m(\Lambda) = s_m(c_1(\Lambda), \dots, c_g(\Lambda)) \in H^{2m}(\mathbb{M}_g; \mathbb{R}),$$

where $c_k(\Lambda)$ is the k-th Chern class of the bundle Λ.

The complex conjugate $\overline{\Lambda}$ satisfies $s_m(\overline{\Lambda}) = (-1)^m s_m(\Lambda)$. Since $\Lambda \oplus \overline{\Lambda}$ is a flat vector bundle on $\mathbb{M}_g$ whose fiber over $[C]$ is the homology group $H_1(C; \mathbb{C})$, we have

$$s_{2n}(\Lambda) = \frac{1}{2} s_{2n}(\Lambda \oplus \overline{\Lambda}) = 0.$$

From the Grothendieck-Riemann-Roch formula or equivalently the Atiyah-Singer index theorem for families, it follows that

$$e_{2n-1} = (-1)^{n-1} \frac{2n}{B_{2n}} s_{2n-1}(\Lambda) \in H^{4n-2}(\mathbb{M}_g; \mathbb{R}). \tag{5.7}$$

Here B_{2n} is the n-th Bernoulli number. In the case $n = 1$ it is exactly the formula (4.2).

Hence the Hodge bundle yields all the odd Morita-Mumford classes, but not the even ones. To get all the Morita-Mumford classes we introduce a higher analogue of the period map, as will be discussed in the succeeding sections.

6. The Earle class and the twisted Morita-Mumford classes

Let Σ_g be a closed oriented C^∞ surface of genus g, $p_0 \in \Sigma_g$ a point, and $v_0 \in T_{p_0}\Sigma_g \setminus \{0\}$ a non-zero tangent vector at the point p_0. We denote by $\mathcal{M}_g$, $\mathcal{M}_{g,*}$ and $\mathcal{M}_{g,1}$ the mapping class groups for the surface Σ_g, the pointed surface (Σ_g, p_0) and the triple (Σ_g, p_0, v_0) respectively. They are the orbifold fundamental groups of the spaces $\mathbb{M}_g$, $\mathbb{C}_g$ and $T^\times_{\mathbb{C}_g/\mathbb{M}_g}$. The fundamental group $\pi_1(\Sigma_g, p_0)$ is naturally embedded into the group $\mathcal{M}_{g,*}$ [40].

By abuse of notation let H denote the real first homology group of Σ_g, $H_1(\Sigma_g; \mathbb{R})$, on which the mapping class groups act in an obvious way. The module H can be interpreted as a flat vector bundle on the moduli space $\mathbb{M}_g$. In 1978 Earle [9] constructed an explicit 1-cocycle $\psi : \mathcal{M}_{g,*} \to H$ such that $(2-2g)\psi$ has values in $H_1(\Sigma_g; \mathbb{Z})$, and $\psi|_{\pi_1(\Sigma_g)}$ is equal to the abelianization map of the group $\pi_1(\Sigma_g)$. Later Morita [35]

independently discovered a cohomology class $k \in H^1(\mathcal{M}_{g,*}; H_1(\Sigma_g;\mathbb{Z}))$ which is equal to $[(2-2g)\psi]$. Furthermore he proved

$$H^1(\mathcal{M}_{g,*}; H_1(\Sigma_g;\mathbb{Z})) = \mathbb{Z}k \cong \mathbb{Z} \tag{6.1}$$

for $g \geq 2$. The author would like to propose the class k should be called *the Earle class*.

The square of the class k is related to the first Morita-Mumford class $e_1 = \kappa_1$ through the intersection pairing

$$m : H \otimes H = H_1(\Sigma_g;\mathbb{R}) \otimes H_1(\Sigma_g;\mathbb{R}) \to \mathbb{R}. \tag{6.2}$$

Morita [36] proved

$$m_*(k^{\otimes 2}) = -e_1 + 2g(2-2g)e \in H^2(\mathcal{M}_{g,*}). \tag{6.3}$$

Here e is the first Chern class of the relative tangent bundle $c_1(T_{\mathbb{C}_g/\mathbb{M}_g}) \in H^2(\mathbb{C}_g) = H^2(\mathcal{M}_{g,*})$.

These phenomena have a higher analogue. The twisted Morita-Mumford class $m_{i,j} \in H^{2i+j-2}(\mathcal{M}_{g,1}; \Lambda^j H)$, $i, j \geq 0$, was introduced in [21]. We have $m_{1,1} = k$ and $m_{i+1,0} = e_i$, $i \geq 1$. All the cohomology classes on the mapping class groups with trivial coefficients (even in the unstable range) obtained from any products of the twisted Morita-Mumford classes by contracting the coefficients using the intersection pairing are exactly the polynomials in the Morita-Mumford classes [25].

This fact is closely related to the Johnson homomorphisms on the mapping class group. The fundamental group $\pi_1(\Sigma_g, p_0, v_0) = \pi_1(\Sigma_g \setminus \{p_0\}, v_0)$ with tangential basepoint v_0 is a free group of rank $2g$. Let Γ_k, $k \geq 0$, denote the lower central series of the free group $\pi_1(\Sigma_g, p_0, v_0)$. We have $\Gamma_0 = \pi_1(\Sigma_g, p_0, v_0)$ and $\Gamma_{k+1} = [\Gamma_k, \Gamma_0]$ for $k \geq 0$. The quotient Γ_1/Γ_2 is naturally isomorphic to $\bigwedge^2 H_1(\Sigma_g;\mathbb{Z}) \subset \bigwedge^2 H$. Let $\mathcal{I}_{g,1}$ be the Torelli group, that is, the kernel of the natural action of $\mathcal{M}_{g,1}$ on the homology group $H_1(\Sigma_g;\mathbb{Z})$. For any $\varphi \in \mathcal{I}_{g,1}$ and $\gamma \in \Gamma_0$, the difference $\gamma^{-1}\varphi(\gamma)$ belongs to Γ_1 from the definition of $\mathcal{I}_{g,1}$. Hence we can define a homomorphism

$$\tau_1(\varphi) : H_1(\Sigma_g;\mathbb{Z}) \to \bigwedge^2 H_1(\Sigma_g;\mathbb{Z}), \quad [\gamma] \mapsto \gamma^{-1}\varphi(\gamma) \bmod \Gamma_2.$$

It is easy to check this induces a homomorphism $\tau_1 : \mathcal{I}_{g,1} \to H^* \otimes \bigwedge^2 H \cong H \otimes \bigwedge^2 H$. The last isomorphism comes from Poincaré duality. Johnson [18] proved the image $\tau_1(\mathcal{I}_{g,1})$ is included in $\bigwedge^3 H$. The homomorphism τ_1 is called the first Johnson homomorphism. Morita [38] proved there exists a unique cohomology class $\tilde{k} \in H^1(\mathcal{M}_{g,1}; \bigwedge^3 H)$ which restricts to τ_1 on the Torelli group $\mathcal{I}_{g,1}$. We call it the extended first Johnson homomorphism. See [40, §7] for more information on the Johnson homomorphisms.

The class $\frac{1}{6}m_{0,3}$ is equal to the extended first Johnson homomorphism $\tilde{k} : \mathcal{M}_{g,1} \to \bigwedge^3 H$ [25]. Each of the Morita-Mumford classes is obtained from some power of $\tilde{k}$ by contracting the coefficients using the intersection pairing m [39]. Conversely for any Sp-module V and any Sp-homomorphism $f : (\bigwedge^3 H)^{\otimes m} \to V$ induced by the intersection pairing, the cohomology class $f_*(\tilde{k}^{\otimes m})$ is a polynomial in the twisted Morita-Mumford class [25]. An extension of the second Johnson homomorphism to the whole mapping class group provides a fundamental relation among the twisted Morita-Mumford classes [22]. In the next section we introduce a flat connection on a vector bundle on the space $T^{\times}_{\mathbb{C}_g/\mathbb{M}_g}$, whose holonomy is an extension of the Johnson homomorphisms to the whole mapping class group $\mathcal{M}_{g,1}$.

7. A higher analgue of the period map

A complex-analytic counterpart of the first Johnson homomorphism is the (pointed) harmonic volume introduced by Harris [17] [46]. It is a real analytic section of a fiber bundle on the moduli $\mathbb{C}_g$ whose fiber over $[C, P_0]$ is $(\bigwedge^3 H_1(C;\mathbb{Z})) \otimes (\mathbb{R}/\mathbb{Z})$. The first variation of the (pointed) harmonic volumes is a twisted 1-form representing the cohomology class $[\tilde{k}]$ [23].

To obtain "canonical" differential forms representing all the twisted Morita-Mumford classes and their higher relations, we construct a higher analogue of the classical period map and the harmonic volume, the harmonic Magnus expansion $\theta : \mathcal{T}_{g,1} \to \Theta_{2g}$ [23]. The space $\mathcal{T}_{g,1} = \widetilde{T^{\times}_{\mathbb{C}_g/\mathbb{M}_g}}$ is Teichmüller space of *triples* (C, P_0, v) of genus g. Here C is a compact Riemann surface of genus g, $P_0 \in C$, and v a non-zero tangent vector of C at P_0 as in §2. For any triple (C, P_0, v) one can define the fundamental group of the complement $C \setminus \{P_0\}$ with the tangential basepoint v denoted by $\pi_1(C, P_0, v)$, which is a free group of rank $2g$. The space Θ_n is the set of all Magnus expansions of the free group F_n of rank $n \geq 2$ in a wider sense stated as follows.

We denote by H the first real homology group of the group F_n, $H_1(F_n;\mathbb{R})$, H^* the first real cohomology group of F_n, $H^1(F_n;\mathbb{R})$, and $[\gamma] \in H$ the homology class of $\gamma \in F_n$. The completed tensor algebra generated by H, $\widehat{T} = \widehat{T}(H) = \prod_{m=0}^{\infty} H^{\otimes m}$, has a decreasing filtration of two-sided ideals $\{\widehat{T}_p\}_{p\geq 1}$ defined by $\widehat{T}_p = \prod_{m\geq p} H^{\otimes m}$. The subset $1 + \widehat{T}_1$ is a subgroup of the multiplicative group of the algebra $\widehat{T}$. We call a map $\theta : F_n \to 1 + \widehat{T}_1$ *a Magnus expansion of the free group F_n* in a wider sense [22], if $\theta : F_n \to 1 + \widehat{T}_1$ is a group homomorphism, and if

$\theta(\gamma) \equiv 1 + [\gamma] \pmod{\widehat{T}_2}$ for any $\gamma \in F_n$. One can endow the set of all Magnus expansions Θ_n with a natural strucure of a (projective limit of) real analytic manifold(s). A certain (projective limit of) Lie group(s) $\mathrm{IA}(\widehat{T})$ acts on Θ_n in a free and transitive way. This induces a series of 1-forms $\eta_p \in \Omega^1(\Theta_n) \otimes H^* \otimes H^{\otimes(p+1)}$, $p \geq 1$, the Maurer-Cartan forms of the action of $\mathrm{IA}(\widehat{T})$, which are invariant under a natural action of the automorphism group of the group F_n, $\mathrm{Aut}(F_n)$. The Maurer-Cartan formula $d\eta = \eta \wedge \eta$ allows us to regard the forms η_p as an equivariant flat connection on the vector bundle $\Theta_n \times H^* \otimes \widehat{T}_2$. The holonomy of the connection is an extension of all the Johnson homomorphisms to the whole group $\mathrm{Aut}(F_n)$. The 1-forms η_p represent the twisted Morita-Mumford classes on the group $\mathrm{Aut}(F_n)$ [22] [23].

Let (C, P_0, v) be a triple of genus g. From now on we denote by H the real first homology group $H_1(C;\mathbb{R})$. As in §5 we denote by $\delta_{P_0} : C^\infty(C) \to \mathbb{R}$, $f \mapsto f(P_0)$, the delta 2-current on C at P_0. Then there exists a $\widehat{T}_1$-valued 1-current $\omega \in \Omega^1(C) \otimes \widehat{T}_1$, satisfying the following 3 conditions

(1) $d\omega = \omega \wedge \omega - I \cdot \delta_{P_0}$, where $I \in H^{\otimes 2}$ is the intersection form.
(2) The first term of ω is equal to $\omega_{(1)} \in \Omega^1(C) \otimes H$ introduced in §5.
(3) $\int_C (\omega - \omega_{(1)}) \wedge *\varphi = 0$ for any closed 1-form φ and each $p \geq 2$.

Using Chen's iterated integrals [8], we can define a Magnus expansion

$$\theta = \theta^{(C,P_0,v)} : \pi_1(C, P_0, v) \to 1+\widehat{T}_1(H_1(C;\mathbb{R})), \quad [\ell] \mapsto 1+\sum_{m=1}^{\infty} \int_\ell \overbrace{\omega\omega\cdots\omega}^{m}.$$

Let a point $p_0 \in \Sigma_g$ and a non-zero tangent vector $v_0 \in T_{p_0}\Sigma_g \setminus \{0\}$ be fixed as in §6. Moreover we fix an isomorphism $\pi_1(\Sigma_g, p_0, v_0) \cong F_{2g}$. A marking α of a triple (C, P_0, v) is an orientation-preserving diffeomorphism of Σ_g onto C satisfying the conditions $\alpha(p_0) = P_0$ and $(d\alpha)_{p_0}(v_0) = v$. For any marked triple $[(C, P_0, v), \alpha]$ we define a Magnus expansion of the free group F_{2g} by

$$F_{2g} \cong \pi_1(\Sigma_g, p_0, v_0) \overset{\alpha_*}{\to} \pi_1(C, P_0, v) \overset{\theta^{(C,P_0,v)}}{\longrightarrow} 1 + \widehat{T}_1(H_1(C;\mathbb{R})) \overset{{\alpha_*}^{-1}}{\to} 1 + \widehat{T}_1.$$

Consequently, the Magnus expansions $\theta^{(C,P_0,v)}$ for all the triples (C, P_0, v) define a canonical real analytic map $\theta : \widetilde{T^\times_{\mathbb{C}_g/\mathbb{M}_g}} = \mathcal{T}_{g,1} \to \Theta_{2g}$, which we call *the harmonic Magnus expansion on the universal family of Riemann surfaces*. The pullbacks of the Maurer-Cartan forms η_p define a flat connection on a vector bundle on the space $T^\times_{\mathbb{C}_g/\mathbb{M}_g}$, and give the canonical differential forms representing the Morita-Mumford classes and their higher relations.

Theorem 7.1 ([23]). *For any $[C, P_0, v, \alpha] \in \mathcal{T}_{g,1}$ we have*

$$(\theta^*\eta)_{[C,P_0,v,\alpha]} = 2\Re(N(\omega'\omega') - 2\omega_{(1)}{}'\omega_{(1)}{}') \in T^*_{[C,P_0,v,\alpha]}\mathcal{T}_{g,1} \otimes \widehat{T}_3.$$

Here $N : \widehat{T}_1 \to \widehat{T}_1$ is defined by $N|_{H^{\otimes m}} = \sum_{k=0}^{m-1} \begin{pmatrix} 1 & 2 & \cdots & m-1 & m \\ 2 & 3 & \cdots & m & 1 \end{pmatrix}^k$, and the meromorphic quadratic differential $N(\omega'\omega')$ is regarded as a $(1,0)$-cotangent vector at $[C, P_0, v, \alpha] \in \mathcal{T}_{g,1}$ in a natural way.

The third homogeneous term $N(\omega'\omega')_{(3)} = N(\omega'_{(1)}\omega'_{(2)} + \omega'_{(2)}\omega'_{(1)})$ is the first variation of the (pointed) harmonic volumes of pointed Riemann surfaces. It represents the extended first Johnson homomorphism $\tilde{k}$. The higher terms provide higher relations among the twisted Morita-Mumford classes. Hence all of the Morita-Mumford classes are represented by some algebraic combinations of $N(\omega'\omega')$.

The second term coincides with $2\omega_{(1)}{}'\omega_{(1)}{}'$, which is exactly the first variation of the period matrices given by Rauch's formula in §5. Hence we may regard the harmonic Magnus expansion as a higher analogue of the classical period map Jac.

8. Secondary objects on the moduli space

The determinant of the Laplacian acting on the space of k-differentials on Riemann surfaces is a 'secondary' object on the moduli space. Zograf and Takhtajan [51] proved that it yields the difference on the moduli space of compact Riemann surfaces, $\mathbb{M}_g$, between a multiple of the Weil-Petersson form ω_{WP} and the Chern form of the Hodge line bundle for the k-differentials induced by the hyperbolic metric. Moreover, they studied analogous phenomena for punctured Riemann surfaces to introduce their Kähler metric, the Zograf-Takhtajan metric, on the moduli space of punctured Riemann surfaces [52].

In this section we discuss other secondary objects, which come from the higher analogue of the period map introduced in §7. Now we can obtain explicit 2-forms from the connection form $N(\omega'\omega')$ on $T^{\times}_{\mathbb{C}_g/\mathbb{M}_g}$, e^J on $\mathbb{C}_g$ and e^J_1 on $\mathbb{M}_g$. Consider the quadratic differential η'_2 defined by

$$\eta'_2 = N(\omega'\omega')_{(4)} \in H^0(C; 2K + 2P_0) \otimes H^{\otimes 4},$$

which satisfies

$$\frac{1}{2g(2g+1)} \mathrm{Res}_{P_0}\left((m \otimes m)(\eta'_2)\right) = -\frac{1}{8\pi^2}.$$

Here m is the intersection pairing $m : H \otimes H \to \mathbb{R}$ as in (6.2). We define

$$e^J = \frac{-2}{2g(2g+1)}\overline{\partial}((m \otimes m)(\eta_2')) \in \Omega^{1,1}(\mathbb{C}_g).$$

From (2.6) e^J represents the first Chern class of the relative tangent bundle

$$[e^J] = e = c_1(T_{\mathbb{C}_g/\mathbb{M}_g}) \in H^2(\mathbb{C}_g; \mathbb{R}).$$

We obtain a twisted 1-form $\eta_1^H \in \Omega^1(\mathbb{C}_g; H)$ representing the Earle class k by contracting the coefficients of $\eta_1' = N(\omega'\omega')_{(3)}$. By (6.3) $m(\eta_1^H)^{\otimes 2} \in \Omega^{1,1}(\mathbb{C}_g)$ represents $-e_1 + 2g(2-2g)e$. So we define

$$e_1^J = -m(\eta_1^H)^{\otimes 2} + 2g(2-2g)e^J$$

which can be regarded as a $(1,1)$-form on $\mathbb{M}_g$ [23, §8].

Hain and Reed [13] already constructed the same form e_1^J in a Hodge-theoretical context. They applied the following lemma to $\frac{1}{12}e_1^J - \mathrm{Jac}^* c_1(\nabla)$ to get a function $\beta_g \in C^\infty(\mathbb{M}_g; \mathbb{R})/\mathbb{R}$, the Hain-Reed function, a secondary object on the moduli space $\mathbb{M}_g$.

Lemma 8.1. *Let M be a connected complex orbifold with $H^0(M; \mathcal{O}) = \mathbb{C}$ and $H^1(M; \mathbb{C}) = H^1(M; \mathcal{O}) = 0$. If a real C^∞ $(1,1)$-form ψ is d-exact, then there exists a real-valued function $f \in C^\infty(M; \mathbb{R})$ such that $\psi = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial} f$. Such a function f is unique up to a constant.*

Here we remark all the holomorphic functions on $\mathbb{M}_g$ are constants provided $g \geq 3$. In fact, each of the boundary component of the Satake compactification of $\mathbb{M}_g$ is of complex codimension ≥ 2. The vanishing of the first cohomology follows from (1.1). See [41]. Hain and Reed also studied the asymptotic behavior of the function β_g towards the boundary of the Deligne-Mumford compactification $\overline{\mathbb{M}_g}^{\mathrm{DM}}$ [13].

We have another 'secondary' phenomenon arround the 2-forms e^J and e_1^J [24]. Let $B = \frac{1}{2g}\omega_{(1)} \cdot \omega_{(1)}$ be the volume form in (5.3). On any pointed Riemann surface (C, P_0) there exists a function $h = h_{P_0} = -\widehat{\Phi}(\delta_{P_0})$ with $d * dh = B - \delta_{P_0}$ and $\int_C hB = 0$. The function $G(P_0, P_1) := \exp(-4\pi h_{P_0}(P_1))$ is just the Arakelov-Green function. We regard G a function on the fiber product $\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g$ and define the $(1,1)$-form e^A on $\mathbb{C}_g$ by

$$e^A := \frac{1}{2\pi\sqrt{-1}}\partial\overline{\partial}\log G|_{\mathrm{diagonal}} \in \Omega^{1,1}(\mathbb{C}_g)$$

representing the Chern class $e = c_1(T_{\mathbb{C}_g/\mathbb{M}_g})$. In fact, the normal bundle of the diagonal map $\mathbb{C}_g \to \mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g$ is exactly the relative tangent bundle $T_{\mathbb{C}_g/\mathbb{M}_g}$.

Furthermore we introduce an explicit real-valued function a_g on $\mathbb{M}_g$ by

$$a_g(C) := \int_C \omega_{(1)} \cdot \widehat{\Phi}(\omega_{(1)} \wedge \omega_{(1)}) \cdot \omega_{(1)}, \tag{8.1}$$

where $\widehat{\Phi}$ is the Green operator introduced in (5.4). By (5.2) we have

$$a_g(C) = -\sum_{i,j=1}^{g} \int_C \psi_i \wedge \overline{\psi_j} \widehat{\Phi}(\overline{\psi_i} \wedge \psi_j). \tag{8.2}$$

We have $a_g(C) > 0$ if $g \geq 2$. Then comparing ∂a_g with η_2' as explicit quadratic differentials, we obtain

$$e^A - e^J = \frac{-2\sqrt{-1}}{2g(2g+1)} \partial\overline{\partial} a_g. \tag{8.3}$$

On the other hand, the integral along the fiber

$$e_1^F := \int_{\text{fiber}} (e^J)^2 \in \Omega^{1,1}(\mathbb{M}_g)$$

also represents the first Morita-Mumford class e_1. By straightforward computation on $\partial\overline{\partial} a_g$ we deduce

Theorem 8.2 ([24])**.**

$$e^A - e^J = \frac{-2\sqrt{-1}}{2g(2g+1)} \partial\overline{\partial} a_g = \frac{1}{(2-2g)^2} ({e_1}^F - {e_1}^J).$$

The function $a_g(C)$ is also a secondary object on the moduli space $\mathbb{M}_g$, and it defines a conformal invariant of the compact Riemann surface C, but the author does not know any of its further properties.

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